Introduction (1)

Inertial particles advected by a turbulent flow are known to have a very inhomogeneous distribution.

Example 1: concentration of light particles (gas bubbles) in vortex dominated zones in a heavier fluid (water).

Example 2: concentration of heavy particles (water droplets) in air, as it occurs in clouds (air). Large fluctuations of the concentration of droplets lead to a higher collision rate; thus to larger bubbles which ultimately leads to precipitation.

Physical origin of the effect: particles are advected by a compressible velocity field, $\mathbf{v}$, even if the turbulent field, $\mathbf{u}$, is incompressible.

Starting point of the analysis:
At low particle concentration, the equation for the concentration of inertial particles obeys:

$$ \partial_t n + \nabla \cdot (\mathbf{v} n) = \kappa \nabla^2 n $$
Introduction (2)

Recently, Balkovsky et al, 2001, proposed a theoretical analysis of the problem of clustering of inertial particles in turbulent flows.

Some important properties of the distribution can be deduced by analysing lagrangian trajectories.

n.b. : The effect of clustering is occurring at small scales (for $r < \eta$, where $\eta$ is the Kolmogorov length scale).

Purpose of the presentation:

With the help of the theoretical analysis of Balkovsky et al, 2001, and of direct numerical simulations (DNS), characterize the coarse-grained density fluctuations at scale $r$, $\overline{n_r}$.

Emphasis here on the quantity:

$$\langle \overline{n_r^2} \rangle$$

as a function of scale, $r$, Reynolds $\#, R_s$, and Stokes $\#, St$, and of the settling velocity.

Application: collision rate of inertial particles, such as droplets in clouds (Falkovich et al, 2002).
Inertial particles in a turbulent flow (1)

Consider particles of radius \( a \), density \( \rho_0 \) in an incompressible fluid of density \( \rho \), viscosity \( \nu \).

Introduce \( \beta = \frac{3\rho}{(\rho + 2\rho_0)} \) and \( \tau_s = \frac{a^2}{3\nu \beta} \). The equation for the particle’s velocity \( \mathbf{v} \) is (Maxey and Riley, 1987):

\[
\begin{bmatrix}
\frac{d\mathbf{v}}{dt} - \beta \frac{d\mathbf{u}}{dt} = (\mathbf{u} - \mathbf{v})/\tau_s + \mathbf{g}
\end{bmatrix}
\]

Case considered here:

- heavy particles (\( \rho_0 \gg \rho \), so \( \beta \ll 1 \)).
- the time \( \tau_s \) is short compared to the Kolmogorov time scale in the flow, \( \tau_K \).

The velocity \( \mathbf{v} \) is:

\[
\mathbf{v} = \mathbf{u} - \tau_s \frac{d\mathbf{u}}{dt} + \tau_s \mathbf{g}
\]

This expression can still be used in the case of a dilute suspension of particles. The velocity field \( \mathbf{v} \) is compressible:

\[
\nabla \cdot \mathbf{v} = -\tau_s \text{tr}(m^2), \quad \text{where} \quad m_{ij} = \frac{\partial u_i}{\partial x_j}
\]

Inertial particles in a turbulent flow (2)

The advection diffusion equation for the particle concentration, \( n \), thus reads:

\[
\frac{Dn}{Dt} \equiv \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = -\nabla \cdot (\nabla n) + \kappa \nabla^2 n
\]

- For inertial particles, \( \nu \gg \kappa \Rightarrow \) the effect of the diffusion term can be neglected down to very small scales, \( \sim \eta(\kappa/\nu)^{1/2} \).

- Starting from an initially uniform concentration, \( n_0 = 1 \), say, concentration fluctuations are produced first at a scale \( \eta \) (experimental evidence: clustering or particles is observed at sizes \( \sim \eta \) (Kostinsky and Shaw, 2001)).

- Along each lagrangian trajectory, neglecting diffusion, the solution of the equation for \( n \) is:

\[
n(t) = n_0 \exp(-\tau_s \int_0^t \text{tr}(m^2)(t') dt')
\]
Inertial particles in a turbulent flow (3)

A parcel of fluid initially at scale $\eta$ gets squashed by the flow, so small scales are generated. After a time $t_r$, which depends on the trajectory, the parcel reaches a small scale $r$.

To estimate the concentration fluctuations accumulated at scale $r$, at a given point $x$ and at a given time $t$, one has to estimate the amount of concentration fluctuations accumulated during the "past", starting at a time $-T_r$, such that a fluid element of size $\eta$ gets squashed to a size $r$ at time $t$.

Inertial particles in a turbulent flow (4)

Method (Falkovich et al, 2002)

- Determine the lagrangian trajectory that arrives at time $t$ and at position $x$.

- Determine the tensor $W_{ij}$, describing the evolution of a line element $\delta l_i(t)$ in the flow: $\delta l_i(t) = W_{ij}(t)l_j(0)$. $W$ satisfies:

$$\frac{dW_{ij}}{dt} = \bar{m}_{ik}\bar{W}_{kj}$$

with $\bar{m}_{ij} = \partial v_i / \partial x_j$.

($\sim$ keep track of the deformation occurring along one trajectory)

- Determine the time $-T_r$ in the past, such that a region of fluid of size $\eta$ starting at $-T_r$ arrives at $t$, at $x$ with a small scale $r$.

$-T_r$ is determined implicitly by the condition that the smallest possible value of $|\delta l(0)| = r$, with the condition that $|\delta l(-T_r)| = \eta$.

- The value of the density at $(x, t)$ is equal to the integral along the lagrangian trajectory:

$$n = n_0 \int_{-T_r}^0 tr(m^2)(t')dt'$$

- The statistical weight of each trajectory is equal to $1/n$. 
Inertial particles in a turbulent flow (5)

In this work, consider the small Stokes number limit: \( St << 1 \), so the difference \(|\mathbf{v} - \mathbf{u}|\) is small.

\[
|\mathbf{v} - \mathbf{u}| \sim \tau_s (\mathbf{u} \cdot \nabla) \mathbf{u} \sim \tau_s / \tau_k |\mathbf{u}| \sim St |\mathbf{u}| << |\mathbf{u}|
\]

Approximations:

- Replace the diffusion equation for the particle density by:
  \[
  \partial_t n + (\mathbf{u} \cdot \nabla) n = - (\nabla \cdot \mathbf{v}) n + \kappa \nabla^2 n
  \]

- Replace the tensor \( \mathbf{m} (= \partial \mathbf{v}) \) by the tensor \( \mathbf{m} (= \partial \mathbf{u}) \) in the evolution equation for \( W \).

Implication of this approximation:

Heavy particles are expelled from vorticity dominated regions. By replacing \( \mathbf{v} \) by \( \mathbf{u} \), particles are allowed to spend more time than they should in vorticity dominated regions.

\( \Rightarrow \) Since \( tr(m^2) = s^2 - \omega^2 / 2 \), this approximation over emphasizes the low density regions.

Inertial particles in a turbulent flow (6)

Effect of the settling velocity:

The gravity term in the velocity induces a settling velocity:

\[
\mathbf{v}_g \equiv \tau_s g
\]

\( \Rightarrow \) compare this term with a typical velocity of the turbulent field.

Use: \( \delta \nu(\eta) \sim \varepsilon^{1/3} \eta^{1/3} \sim (\varepsilon \nu)^{1/4} \), where \( \varepsilon \) is the rate of dissipation of kinetic energy and \( \nu \) the viscosity, and define:

\[
\varepsilon = \frac{(\varepsilon \nu)^{1/4}}{|\mathbf{v}_g|}
\]

Physical situation considered here: a population of (water) droplets in air. Particles differ only by their radii, \( a \)

\[
St \propto a^2, \quad |\mathbf{v}_g| \propto a^2 \Rightarrow St \times \varepsilon = \varepsilon_0
\]

Here, study the influence of the settling velocity keeping the product \( \varepsilon_0 \) constant.
Numerical study (1)

Direct Numerical Simulation of turbulent flow:

Generate an Eulerian flow in a 3-dimensional periodic box. Use a pseudo-spectral code to solve the Navier Stokes equations:

\[
\partial \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} ; \quad \nabla \cdot \mathbf{u} = 0
\]

Resolution: up to \(256^3\) grid points, \(R_\lambda \leq 130\).
Maintain adequate spatial resolution \((k_{\text{max}} \eta \geq 1.5)\).

Lagrangian aspects (Yeung and Pope, 1988)

Once the Eulerian flow is known, determine the velocity at a set of Lagrangian points \(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n\) by interpolating the velocity fields from the numerical lattice to the particle’s positions (cubic splines).

Evolve the lagrangian markers according to

\[
\frac{d\vec{x}_i}{dt} = \vec{v}_i
\]

Use a predictor corrector algorithm, second order accurate in time.

Efficient and accurate way to track numerically lagrangian particles.

Numerical study (2)

A difficulty: It is in practice impossible to follow lagrangian trajectories backwards in time.

Here, we follow lagrangian trajectories forward in time, and monitor \(W^{-1}\), the inverse of \(W\):

\[
\frac{dW^{-1}_{ij}}{dt} = -W^{-1}_{ik} m_{kj}
\]

Because of the relation between the fluid elements at time \(t = 0\) and at time \(T > 0\): \(\delta l_i(0) = W^{-1}_{ij}\delta l_j(T)\), the compression of the line element \(\delta l\) between time \(t = 0\) and time \(t = T\) is equivalent to a growth of \(W^{-1}\).

\(\Rightarrow\) simply monitor the growth of \(|W^{-1}|\)!

Numerical algorithm:

- Generate a statistically steady turbulent flow.
- Release a set of lagrangian particles at \(t = 0\); follow them in time and monitor \(W^{-1}(t = 0) = \delta_{ij}\).
- At the first time, \(t_r\), such that \(|W^{-1}(t_r)|/|W^{-1}(0)|\) reaches the ratio \(\eta/r\), compute the integral:

\[
I_r = \int_0^{t_r} tr(m^2)(t')dt'
\]

Get statistics over \(\sim 5.10^5 - 5.10^6\) trajectories.
Statistics of stretching (1)

Asymptotically, \(|W^{-1}|\) grows as:

\[ |W^{-1}| \sim \exp(-\lambda_3 t) \]

where \(\lambda_3\) is the smallest Lyapunov exponent of the flow.

Question: value of the smallest Lyapunov exponent?

Compute the statistics of the compression rate, \( r_c \equiv \ln(\delta l(0)/\delta l(t)) \)
induced by the flow.

Numerical observations:

- \(\langle r_c \rangle\) grows linearly with time.
  
  \[ \Rightarrow \text{extract the third Lyapunov exponent, } -\lambda_3 \]

- In the absence of settling velocity (\(\epsilon_0 = \infty\)),
  
  \[ \lambda_3 \tau_K \approx -0.16 \approx -1/6 \]

  (consistent with Girimaji and Pope, 1991).

- At a fixed time, the compression rate \( r_c \) has a pdf independent of time, as a first approximation. The distribution is slightly non-gaussian.
Statistics of stretching (2).

Consequences:

* In the absence of a settling velocity, the time \( t_r \) it takes for \( |W^{-1}| \) to grow to a value \( \eta/r \) behaves like:

\[
\langle t_r \rangle \approx 6\tau_K \ln(\eta/r)
\]

* The value of \( t_r \) fluctuates with a rms growing as a function of \( r \) slower than \( \langle t_r \rangle \).

Practical implications:

The Navier-Stokes equations are integrated for a duration \( T_s \). To obtain reliable results up to \( \eta/r \gtrsim 256 \), a value of \( T_s \approx 50 - 60\tau_K \) was found to be sufficient.

In our runs, in the absence of a settling velocity (\( \epsilon_0 = \infty \)):

- \( T_s/\tau_K \approx 60 - 65 \) for \( R_\lambda \leq 105 \).
- \( T_s/\tau_K \approx 50 \) for \( R_\lambda = 130 \).
Statistics of stretching (3)

Effect of a settling velocity:

- At a fixed value of $\epsilon$, the stretching rate grows monotonically with $R_\lambda$.

- At fixed Reynolds number, the stretching rate decreases when the settling velocity increases (when $\epsilon$ diminishes).

- Our numerical data are consistent with a dependence of the stretching rate of the form:

\[ -\lambda_3 \tau_K = F \left( \frac{\langle u^2 \rangle^{1/2}}{|v_g|} \right) \]

where $\langle u^2 \rangle^{1/2}$ is the $rms$ of the turbulent velocity fluctuations.

Consequence: to reach a small scale requires a longer integration time with a settling velocity, than when $|v_g| = 0$. 
The second order moment $\langle \eta^2 \rangle$ has roughly a power law dependence as a function of $\eta/\tau$.

At a fixed value of the Reynolds number, $\langle \eta^2 \rangle$ grows faster when the Stokes number increases.

When the settling velocity is small, the growth of $\langle \eta^2 \rangle$ becomes stronger when the Reynolds number increases.

small-scale turbulence does affect the distribution of particles.
Structure of the second order moment \( \langle \eta^2 \rangle \) (2)

Fit of the curve \( \langle \eta^2 \rangle \) vs. \( \eta/r \):

- Compute the local exponent

\[
\alpha \equiv \frac{d \ln \langle \eta^2 \rangle}{d \ln (\eta/r)}
\]

(numERICALLY, take finite differences of the data).

- Observe that the local exponent \( \alpha \) is well defined, and relatively constant over the range \( 50 \leq \eta/r \leq 200 \), with possibly a very slight linear dependence.

- Plot the value of the exponent \( \alpha \) as a function of \( R_\lambda \) and \( St \).

\[ \alpha \text{ increases both as a function of } St \text{ and } R_\lambda \]
Dr. Alain Pumir, CNRS, INLN (KITP Pattern Formation 10-23-03) Intermittent Distribution of Heavy Particles in a Turbulent Flow
Structure of the second order moment \( \langle \vec{v}^2 \rangle \) (3)

In the absence of a settling velocity, the value of the exponent \( \alpha \) can be very well fitted as a function of \( \alpha \) by the following functional form:

\[
\alpha(R_\lambda, St) \approx a_1(R_\lambda) \times St + a_2(R_\lambda) \times St^2
\]

with:

- \( a_1(R_\lambda) \approx 0.4 \)
- \( a_2(R_\lambda) \) increases as a function of \( R_\lambda \).

Theoretical challenge (... with some potential interesting applications): relate these values to other quantities characterizing the turbulent flow itself.
Structure of the second order moment $\langle \bar{\pi}^2 \rangle$ (4)

Influence of the settling velocity.

The effect of a finite settling velocity is to reduce the level of fluctuations of the $\langle \bar{\pi}^2 \rangle$.

- At $\epsilon_0 = 0.06$, the exponent is still a growing function of $R_{\lambda}$ and $St$.

- At larger value of the settling velocity, $\epsilon_0 = 0.02$, the exponent is a growing function of $St$, but hardly grows with $R_{\lambda}$. 
Conclusion

The density of particles coarse grained over a volume of size \( r \), \( \bar{n}_r \) has been studied, by using

- \((i)\) the theoretical formulation of Balkovski et al, 2001 (see also Falkovich et al, 2002), which expresses \( \bar{n}_r \) in terms of the integral of \( \nabla \cdot \mathbf{v} \) along lagrangian trajectories.

- \((ii)\) direct numerical simulations of particles at moderate Reynolds numbers \( (R_\lambda \leq 130) \).

Results here mostly on the second moment \( \langle \bar{n}_r^2 \rangle \).

- **Existence of a power law dependence**

\[
\langle \bar{n}_r^2 \rangle \propto (\eta/r)^\alpha
\]

- The exponent \( \alpha \) depends on Reynolds number \( R_\lambda \) and on the Stokes number \( St \), and on the settling velocity (through the parameter \( e \)).

- In the absence of settling velocity, the level of fluctuation is **very high**.

- The settling velocity plays a subtle role. In any event, it strongly **reduces** the level of fluctuations of \( \bar{n}_r \).